

GRÜN'S THEOREMS AND CLASS GROUPS

FRANZ LEMMERMEYER

ABSTRACT. In this article we show how Grün's results in group theory can be used for studying the structure of class groups in normal extensions.

Otto Grün, who is known for his contributions to group theory, was an amateur mathematician; for details on his "remarkable career" and his correspondence with Hasse, see Roquette's article [8]. Grün's idea of studying the action of the Galois group on class groups led him into problems in representation theory and group theory. In this article, I would like to explain how Grün's results can be applied to class field theory.

1. GRÜN'S PROBLEM 153

Grün's excursions into group theory started with the following observation¹, which Grün communicated to Hasse in a letter dated Dec. 19, 1932:

Proposition 1. *Let L/K be a normal extension with degree n and Galois group G , and assume that the class group $\text{Cl}(L)$ of L is cyclic. If G equals its commutator subgroup G' , then $c^n = 1$ for all $c \in \text{Cl}(L)$.*

Grün did not trust his results and asked Hasse whether he could find an error. Hasse could not, but suggested to ask Olga Taussky whether the result was known. Taussky sent Arnold Scholz a counterexample, which Scholz showed not to be valid; in addition he proved a large part of Grün's claim by observing that the action of a group G on a cyclic group is abelian. Eventually Hasse suggested posing Grün's result as a problem in the Jahresberichte der DMV. By then, Holzer had given his result the following form:

Proposition 2. *Let L/k be a normal extension of number fields with Galois group G , and let K/k be the maximal abelian subextension of L/k . If $\text{Cl}(L)$ is cyclic, then $h_L \mid (L : K)h_K$.*

2. GALOIS ACTION

Results such as Grün's should be seen as variations of a now classical theme: the action of a group G on a finite abelian group A puts constraints on the structure of A . In fact, define the relative class group $\text{Cl}(K/k)$ of a normal extension K/k of number fields by the exact sequence

$$1 \longrightarrow \text{Cl}(K/k) \longrightarrow \text{Cl}(K) \xrightarrow{N} \text{Cl}(k)$$

induced by the norm map $N : \text{Cl}(K) \longrightarrow \text{Cl}(k)$. Then we have

1991 *Mathematics Subject Classification.* 11R29.

Key words and phrases. ideal class group; Galois group.

¹Grün used the rationals as his base field, but everything is valid over an arbitrary number field.

Proposition 3. *Let K/k be a cyclic extension of prime degree n . For a prime $p \nmid n$, let f denote the order of $p \bmod n$. Then the p -rank of $\text{Cl}(K/k)$ is divisible by f .*

The first special case of this proposition was found by Kummer, who used a similar technique for showing that the class group of $\mathbb{Q}(\zeta_{29})$ has type $(2, 2, 2)$. For proofs of more general results in this direction see [5]; the essential lines in the historical development were described by Metsänkylä [6].

Corollary 4. *If $\text{Cl}_p(K)$ has rank $< f$, then it comes from k .*

We say that a piece C of the class group $\text{Cl}(K)$ comes from a subfield k if the norm map $N : C \rightarrow \text{Cl}(k)$ is injective.

Proof. Prop. 3 implies that $\text{Cl}_p(K/k)$ is trivial, hence the norm map $\text{Cl}_p(K) \rightarrow \text{Cl}_p(k)$ is injective. \square

Grün's first attempt at generalizing his results on cyclic class groups was the following theorem, which he announced in a letter to Hasse dated Dec. 5, 1933:

Theorem 5. *Let G be a finite group acting on an elementary-abelian p -group A , and let m denote the rank of A . For a prime $\ell \neq p$ let m_ℓ denote the order of $p \bmod \ell$, and let P denote an ℓ -Sylow subgroup of G .*

If $\ell > \frac{m}{m_\ell}$, then P' acts trivially on A ; that is, the ℓ -Sylow subgroups of the automorphism group of A are abelian.

Grün's Thm. 5 does not have nontrivial applications for $\ell = 2$, since in this case we have $m_2 = 1$, and the result applies only if $2 > m$, i.e., if $\text{Cl}_2(K)$ is cyclic. For applications to p -class groups for $p \geq 3$, see Section 3 below. In his article [3], Grün removed the restriction that A have exponent p , and showed

Theorem 6. *Let A be an abelian p -group on which a finite group G acts. If ν is the smallest exponent with $\ell^\nu > \frac{m}{m_\ell}$, then the ν -th commutator subgroup of an ℓ -Sylow subgroup of G acts trivially on A .*

Grün's proof is based on the structure of $G_n = \text{GL}_n(\mathbb{F}_q)$, which he had obtained in [2]. There he started by observing that

$$\#G_n = (p^{nf} - 1)(p^{nf} - p^f) \cdots (p^{nf} - p^{(n-1)f}),$$

where $q = p^f$, defined m_ℓ as the minimal positive integer with $p^{f m_\ell} \equiv 1 \bmod \ell$, and then proved

Theorem 7. *The ℓ -Sylow subgroups of $\text{GL}_n(\mathbb{F}_q)$ are elementary abelian ℓ -groups of order ℓ^{r_i} if $\left\lfloor \frac{n}{m_\ell} \right\rfloor = r < \ell$ and $\ell^i \parallel (p^{f m_\ell} - 1)$.*

They are $(r+1)$ -stage metabelian groups if

$$\ell^r \leq \left\lfloor \frac{n}{m_\ell} \right\rfloor < \ell^{r+1}.$$

Grün next claims ([3, Satz 1])

Theorem 8. *Assume that G is a finite group, and that $A \leq G$ is an abelian p -group of p -rank m and normal in G ; let L denote an ℓ -Sylow subgroup of G for some prime $\ell \neq p$, and let m_ℓ denote the order of $\ell \bmod p$.*

If k is minimal with $\frac{m}{m_\ell} < \ell^k$, then A commutes with the k -th commutator group $L^{(k)}$. In other words: $L^{(k)}$ acts trivially on A .

For the proof of Theorem 8, Grün considers the map $\rho : G \longrightarrow \text{Aut}(A)$ and assumes first that A is elementary abelian. Then $\text{im } \rho$ can be identified with some subgroup of $\text{GL}_m(p)$. Let N be the group of all elements of G that commute with A . Then N is normal with $G/N \simeq \text{im } \rho \subseteq \text{GL}_m(p)$. By Thm. 7, the k -th commutator group of the ℓ -Sylow of G/N is trivial. But the ℓ -Sylow subgroups of G/N have the form LN/N , where L is an ℓ -Sylow subgroup of G , and we have $(LN/N)^{(k)} \simeq L^{(k)}N/N$; this shows that $L^{(k)} \subseteq N$, which means that $L^{(k)}$ acts trivially on A . The claim for general abelian p -groups A is proved by induction.

3. APPLICATIONS

In this section we show how Grün's results can be applied to obtaining results on ideal class groups in normal extensions. The general setup is this: L/k is a normal extension with Galois group G ; this group G acts on $\text{Cl}(L)$ or on certain subgroups; if some subgroup of G acts trivially, parts of the class group must come from proper subfields of L .

In fact, let L/k be a nonabelian ℓ -extension with Galois group G , and let K be the fixed field of G' . Let m denote the rank of the p -class group of L . Grün's Theorem 5 shows:

- (1) If $p \equiv 1 \pmod{\ell}$ and $m = 1$, then $\text{Cl}_p(L)$ comes from K .
- (2) If $p \equiv -1 \pmod{\ell}$ ($\ell \neq 2$) and $m \leq 2$, then $\text{Cl}_p(L)$ comes from K .
- (3) If $p \not\equiv \pm 1 \pmod{\ell}$ and $m \leq 3$, then $\text{Cl}_p(L)$ comes from K .

Consider e.g. an extension L/k whose Galois group G is one of the two non-abelian groups of order ℓ^3 . Its commutator subgroup G' is cyclic of order ℓ , and $G/G' \simeq (\ell, \ell)$. Let K be the fixed field of G' ; then K/k is the maximal abelian subextension of L/K , and $\text{Gal}(K/k) \simeq (\ell, \ell)$.

If $A = \text{Cl}_p(L)$ has p -rank m , and if m_ℓ denotes the order of $p \pmod{\ell}$, then G' acts trivially if $\ell > m/m_\ell$. Thus if $\text{Cl}_p(L)$ has p -rank at most 2, then it must come from K . Stronger results follow if there are nontrivial lower bounds for m_ℓ : unless $p \equiv 1 \pmod{\ell}$, we have $m_\ell \geq 2$, and the conclusion above holds as soon as $2\ell > m$.

If $\ell = 3$ and $p \equiv 2 \pmod{3}$, then $\ell > m/m_3$ is satisfied whenever $m < 6$. Thus if $\text{rank } \text{Cl}_2(L) < 6$, then $G = \text{Gal}(L/\mathbb{Q})$ acts in such a way on the p -elementary part of the class group that the commutator group acts trivially.

Proposition 9. *Let L/\mathbb{Q} be a normal extension whose Galois group is isomorphic to one of the two nonabelian groups of order ℓ^3 . If $p \equiv 2 \pmod{3}$ is prime and if the rank of $\text{Cl}_p(L)$ is ≤ 5 , then $\text{Cl}_p(L)$ comes from the maximal abelian subfield K of L/\mathbb{Q} . If $p \equiv 1 \pmod{3}$ and the rank of $\text{Cl}_p(L)$ is ≤ 2 , then $\text{Cl}_p(L)$ comes from the maximal abelian subfield K of L/\mathbb{Q} .*

We have already remarked that Grün's Thm. 5 does not apply to 2-extensions. But we can apply Thm. 6 to 2-class field towers of sufficiently high stage. Consider e.g. the 2-class field towers of complex quadratic number fields k with class group $(2, 2, 2)$ studied e.g. in [1], [4], and [7]. If k^3 has a class number divisible by an odd prime p , then either it comes from k^2 , or the p -rank of $\text{Cl}(k^3)$ is large:

Proposition 10. *Let k be a quadratic number field whose 2-class field tower $k \subset k^1 \subset k^2 \subset \dots$ has at least 3 steps, and let $G = \text{Gal}(k^3/k)$ denote the Galois group of the 2-extension k^3/k . If $C = \text{Cl}_p(k^3)$ has rank ≤ 3 , then G'' acts trivially on C , and $\text{Cl}_p(k^2)$ has a subgroup isomorphic to C .*

REFERENCES

- [1] E. Benjamin, F. Lemmermeyer, C. Snyder, *Imaginary quadratic fields with $\text{Cl}_2(k) = (2, 2, 2)$* , J. Number Theory **103** (2003), 38–70
- [2] O. Grün, *Über Substitutionsgruppen im Galoisfeld $G(p^f)$* , J. Reine Angew. Math. **171** (1934), 170–172
- [3] O. Grün, *Beiträge zur Gruppentheorie. I*, J. Reine Angew. Math. **174** (1935), 1–14
- [4] F. Lemmermeyer, *On the 2-class field towers of some imaginary quadratic number fields*, Abh. Math. Sem. Hamburg **67** (1997), 205–214
- [5] F. Lemmermeyer, *Galois action on class groups*, J. Algebra **264** (2003), 553–564
- [6] T. Metsänkylä, *On the history of the study of ideal class groups*, Expo. Math. **25** (2007), 325–340
- [7] H. Nover, *Computation of Galois groups associated to the 2-class towers of some imaginary quadratic fields with 2-class group $C_2 \times C_2 \times C_2$* , J. Number Theory **129** (2009), 231–245
- [8] P. Roquette, *From FLT to finite groups. The remarkable career of Otto Grün*, Jahresber. DMV **107** (2005), 117–154

MÖRIKEWEG 1, 73489 JAGSTZELL, GERMANY
E-mail address: `hb3@ix.urz.uni-heidelberg.de`